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# Global Glacial Isostatic Adjustment: Target Fields for Space Geodesy

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## Abstract

*A very detailed theory of the global process of glacial isostatic adjustment (GIA) is now available that is being employed to address a number of significant problems in both solid Earth geophysics and climate dynamics. A recent focus of the work in this area has been upon the impact of changes in the Earth's rotational state upon postglacial sea level history and the modern field of geoid height time dependence that is being measured by the GRACE dual satellite system that is now in space. Satellite laser ranging continues to play a critical role in the understanding of these processes. This paper summarizes recent progress in modelling the impact of the GIA process upon Earth's rotational state.*

## Introduction

The origins of highly significant anomalies in the Earth's rotational state, respectively the so-called non-tidal acceleration of the rate of axial rotation and the secular drift (true polar wander) of the pole of rotation relative to the surface geography, have been associated for some time with the influence of the glacial isostatic adjustment (GIA) process. The non-tidal acceleration is equivalent to a value for the time dependence of the degree 2 zonal coefficient in the spherical harmonic expansion of Earth's gravitational field, commonly represented as  $\dot{J}_2$  of  $(-2.67 \pm 0.15) \times 10^{-11}$  year<sup>-1</sup> (e.g. Cheng et al. 1989). The value for the rate of polar wander reported by Vincente and Yumi (1969, 1970) using the data of the International Latitude Service (ILS) was  $(0.95 \pm 0.15)$  degree/million years, a value that is close to the most recent estimation by Argus and Gross (2004) of 1.06 degree/million years. The latter authors have suggested that the observed direction and speed of polar wander should be corrected for the influence of plate tectonic motions and that this could be a significant effect, depending upon the assumptions on the basis of which the correction is made (see Table 1 of Argus and Gross, 2004).

The development of theoretical explanations for the above discussed anomalies in Earth rotation has been dominated by work over the past two decades that has suggested a close connection of them both to GIA. The earliest discussion of the impact upon polar wander that should be expected due to time dependent surface loading of a visco-elastic model of the Earth was that of Munk and MacDonald (1960) who employed a simple homogeneous model to suggest that wander of the pole could only occur in response to simultaneous variability in the surface mass load. This point was obscured in the later papers by Nakiboglu and Lambeck (1980, 1981) and Sabadini and Peltier (1981) whose analysis was based upon the application of a homogeneous viscoelastic model similar to that employed by Munk and MacDonald (1960). These authors, however, suggested that polar wander would continue on a homogeneous visco-elastic model of the Earth even after all temporal variations of the surface mass load had ceased. This significant error of interpretation was corrected in Peltier (1982) and Wu and Peltier (1984) who showed that, in the case of cyclic loading and unloading, as is appropriate for the computation of the GIA

effect following the series of glacial loading and unloading events that have characterized the Late Quaternary period of Earth history (e.g. Broecker and van Donk, 1970), there would be no polar wander effected once the cycle ended. The homogeneous visco-elastic model of the planet would therefore exhibit no memory of the past history of loading and unloading as correctly pointed out by Munk and McDonald. This was traced to the fact that, specifically for the homogeneous visco-elastic model, there exists an exact annihilation of the polar wander forced by the internal redistribution of mass due to the free relaxation of Earth's shape and that forced by the deformation due to the changing rotation itself (see e.g. Figure 2 of Wu and Peltier 1984).

Based upon the prior analysis of Peltier (1974, 1976), however, it was known that realistic viscoelastic models of the planetary interior were significantly more complex than could be accommodated by the homogeneous visco-elastic model of Munk and MacDonald (1960). Whereas the relaxation under surface forcing of a homogeneous visco-elastic model of the Earth is described by a single relaxation time that is unique for each spherical harmonic degree in the deformation spectrum, realistically layered spherical visco-elastic models have a much more complex relaxation spectrum, a unique spectrum consisting of an (often essentially) finite number of modes for each spherical harmonic degree. In Peltier (1982) and Wu and Peltier (1984) it was demonstrated that this realistic level of complexity endowed the Earth model with a memory of its history of surface loading and unloading such that the pole of rotation would continue to wander even after the surface load had ceased to vary. Deep sea core oxygen isotopic data based upon  $\delta^{18}O$  measurements on benthic foraminifera were employed as basis for the construction of a model of cyclic ice-sheet loading and unloading of the continents following the interpretation of such data as proxy for the variation of continental ice volume through time (Shackleton 1967, Shackleton and Opdyke 1973). Analysis based upon the application of rather crude models of the growth and decay of the Laurentide, Fennoscandian and Antarctic ice sheets then demonstrated that both the speed and direction of true polar wander as well as the non-tidal acceleration of rotation could be fit by the model and that the radial visco-elastic structure required to fit both of these observations was essentially the same. This was construed to strongly suggest that both anomalies might to be entirely explained as a consequence of the ongoing global process of glacial isostatic adjustment.

A recent objection to this interpretation was raised in the paper by Mitrovica, Wahr et al. (2005; hereafter MW) who have suggested that the theoretical formulation employed in Peltier (1982) and Wu and Peltier (1984) was mathematically "unstable" insofar as the computation of the polar wander component of the response to the GIA process is concerned. This objection appears to be based upon an error of mathematical comprehension as explicit analyses to be presented in what follows will demonstrate.

### **Computation of the rotational response of the Earth to the GIA process**

The time dependent impact on the Earth's rotational state of the glacial isostatic adjustment process is determined as a solution of the classical Euler equation describing the conservation of angular momentum of a system subjected to no external torques, as:

$$\frac{d}{dt} (J_{ij} \omega_i) + \epsilon_{ijk} \omega_j J_{k\ell} \omega_\ell = 0 \quad (1)$$

in which the  $J_{ij}$  are the elements of the moment of inertia tensor, the  $\omega_i$  are as previously and  $\epsilon_{ijk}$  is the Levi-Civita (alternating) tensor. Restricting attention to small departures from the modern state of steady rotation with angular velocity  $\Omega_o$ , we may construct a solution to (1), accurate to first order in perturbation theory, by expanding:

$$\omega_i = \Omega_o (\delta_{i3} + m_i); m_i = \omega_i / \Omega_o \quad (2a)$$

$$J_{11} = A + I_{11} \quad (2b)$$

$$J_{22} = B + I_{22} \quad (2c)$$

$$J_{33} = C + I_{33} \quad (2d)$$

$$J_{ij} = I_{ij}, i \neq j \quad (2e)$$

Substitution of these expansions into equation (1), keeping only terms of first order, leads to the standard set of governing equations for polar wander and the length of day, respectively (see Munk and McDonald, 1960), as:

$$\left. \begin{aligned} \frac{dm_1}{dt} + \frac{(C-B)}{A} \Omega_o m_2 &= \Psi_1 \\ \frac{dm_2}{dt} + \frac{(C-A)}{B} \Omega_o m_1 &= \Psi_2 \end{aligned} \right\} \text{polar wander} \quad (3a,b)$$

$$\left. \frac{dm_3}{dt} = \Psi_3 \right\} \text{length of day} \quad (3c)$$

in which the ‘‘excitation functions’’ are defined as:

$$\Psi_1 = \left( \frac{\Omega_o}{A} \right) I_{23} - \frac{(dI_{13}/dt)}{A} \quad (4a)$$

$$\Psi_2 = - \left( \frac{\Omega_o}{B} \right) I_{13} - \frac{(dI_{23}/dt)}{B} \quad (4b)$$

$$\Psi_3 = - \left( \frac{I_{33}}{C} \right) \quad (4c)$$

Now it is critical to recognize that there exist perturbations  $I_{ij}$  to the inertia tensor due to two distinct causes, namely due to the direct influence of change in the mass distribution of the planet that accompanies the change in planetary shape due to surface loading and unloading and that due to the additional deformation induced by the changing rotation triggered by the surface mass loading and unloading process. The contribution due to the former process may be represented as (e.g. Peltier, 1982):

$$I_{ij}^{GIA} = \left( 1 + k_2^L(t) \right) * I_{ij}^R(t) \quad (5)$$

in which  $k_2^L(t)$  is the surface mass load Love number of degree 2 and the  $I_{ij}^R$  are the perturbations of inertia that would obtain due to the variation in surface mass load if the Earth were rigid. The symbol \* in equation (5) represents the convolution operation. The contribution to the perturbations of inertia due to the changing rotation

follows from an application of a linearized version of MacCullagh's formula (e.g. see Munk and MacDonald, 1960) as:

$$I_{13}^{\text{ROT}} = \left( \frac{k_2^T * a^5 \omega_1 \omega_3}{3G} \right) = \left( \frac{k_2^T}{k_f} \right) * m_1 (C - A) \quad (6a)$$

$$I_{23}^{\text{ROT}} = \left( \frac{k_2^T * a^5 \omega_2 \omega_3}{3G} \right) = \left( \frac{k_2^T}{k_f} \right) * m_2 (C - A) \quad (6b)$$

with 
$$k_f = \left( \frac{3G}{a^5 \Omega_0^2} \right) (C - A) \quad (6c)$$

the value of which is determined entirely by the observed flattening of the Earth's figure. Assuming the validity of the data in Yoder (1995) as listed on the web site: ([www.agu.org/references/geophys/4\\_Yoder.pdf](http://www.agu.org/references/geophys/4_Yoder.pdf)), one obtains the value  $k_f \cong 0.9414$ , a value that deviates somewhat from the value of 0.9382 employed in MW.

### The General Solution for the Rotational Response in the Laplace Transform Domain

Since the solution of equation (3c) for the change in the axial rate of rotation is uncomplicated, it will suffice to focus first in what follows on the solution of (3a) and (3b) for the polar wander component of the response to surface loading. Substitution of (6a) and (6b) into (3a,b), the Laplace-transformed forms of the equations that follow are simply:

$$s m_1 + \sigma \left( 1 - \frac{k_2^T(s)}{k_f} \right) m_2 = \Psi_1(s) \quad (7a)$$

$$s m_2 + \sigma \left( 1 - \frac{k_2^T(s)}{k_f} \right) m_1 = \Psi_2(s) \quad (7b)$$

where 
$$\sigma = \Omega_0 \frac{(C - A)}{A} \quad (7c)$$

is the Chandler Wobble frequency of the rigid Earth, "s" is the Laplace transform variable, and again A=B has been assumed. The Laplace-transformed forms of the excitation functions in (4a) and (4b) are simply:

$$\Psi_1(s) = \left( \frac{\Omega_0}{A} \right) I_{23}(s) - \left( \frac{s}{A} \right) I_{13}(s) \quad (8a)$$

$$\Psi_2(s) = \left( \frac{\Omega_0}{A} \right) I_{13}(s) - \left( \frac{s}{A} \right) I_{23}(s) \quad (8b)$$

with 
$$I_{ij}(s) = \left( 1 + k_2^L(s) \right) I_{ij}^{\text{Rigid}}(s) \quad (8c)$$

Now equations (7a) and (7b) are elementary algebraic equations for  $m_1(s)$  and  $m_2(s)$  and these may be solved exactly to write:

$$m_1(s) = \frac{1 + k_2^L(s)}{s^2 + \sigma^2 \left(1 + \frac{k_2^T(s)}{k_f}\right)^2} \left[ \left( \frac{\Omega_o \sigma}{A} \right) \left( 1 - \frac{k_2^T(s)}{k_f} \right) - \frac{s^2}{A} \right] I_{13}^{\text{Rigid}}(s) \quad (9a)$$

$$m_2(s) = \frac{1 + k_2^L(s)}{s^2 + \sigma^2 \left(1 + \frac{k_2^T(s)}{k_f}\right)^2} \left[ \left( \frac{\Omega_o \sigma}{A} \right) \left( 1 - \frac{k_2^T(s)}{k_f} \right) - \frac{s^2}{A} \right] I_{23}^{\text{Rigid}}(s) \quad (9b)$$

If we now neglect terms of order  $s^2/\sigma^2$  in (9a,b), which delivers a highly accurate approximation free of the influence of the Chandler wobble, we obtain:

$$m_1(s) = \left( \frac{\Omega_o}{A \sigma} \right) \frac{1 + k_2^L(s)}{1 - \frac{k_2^T(s)}{k_f}} I_{13}^{\text{Rigid}}(s) = H(s) I_{13}^{\text{Rigid}}(s) \quad (10a)$$

$$m_2(s) = H(s) I_{23}^{\text{Rigid}}(s) \quad (10b)$$

A convenient short-hand form for the solution vector  $(m_1, m_2) = \underline{m}$  is to write:

$$\underline{m}(s) = \frac{\underline{\Psi}^L(s)}{\begin{bmatrix} k_2^T(s) \\ 1 - \frac{k_2^T(s)}{k_f} \end{bmatrix}} = H(s) \left( I_{13}^{\text{Rigid}}(s), I_{23}^{\text{Rigid}}(s) \right) \quad (11a)$$

$$\text{where } \underline{\Psi}^L(s) = \left[ \left( \frac{\Omega_o}{A \sigma} \right) (1 + k_2^L(s)) \left( I_{13}^{\text{Rigid}}(s), I_{23}^{\text{Rigid}}(s) \right) \right] \quad (11b)$$

### An Exact Inversion of the Laplace Transform Domain Solution

From equations (11) it will be clear that the polar wander solution  $m(s)$  will depend critically upon the ratio  $k_2^T(s)/k_f$ . This fact was more fully exposed in the analysis of Peltier (1982) and Wu and Peltier (1984) who re-wrote the Laplace transform domain forms of  $k_2^T(s)$  and  $k_2^L(s)$  as (e.g. see equation 61 of Wu and Peltier 1984):

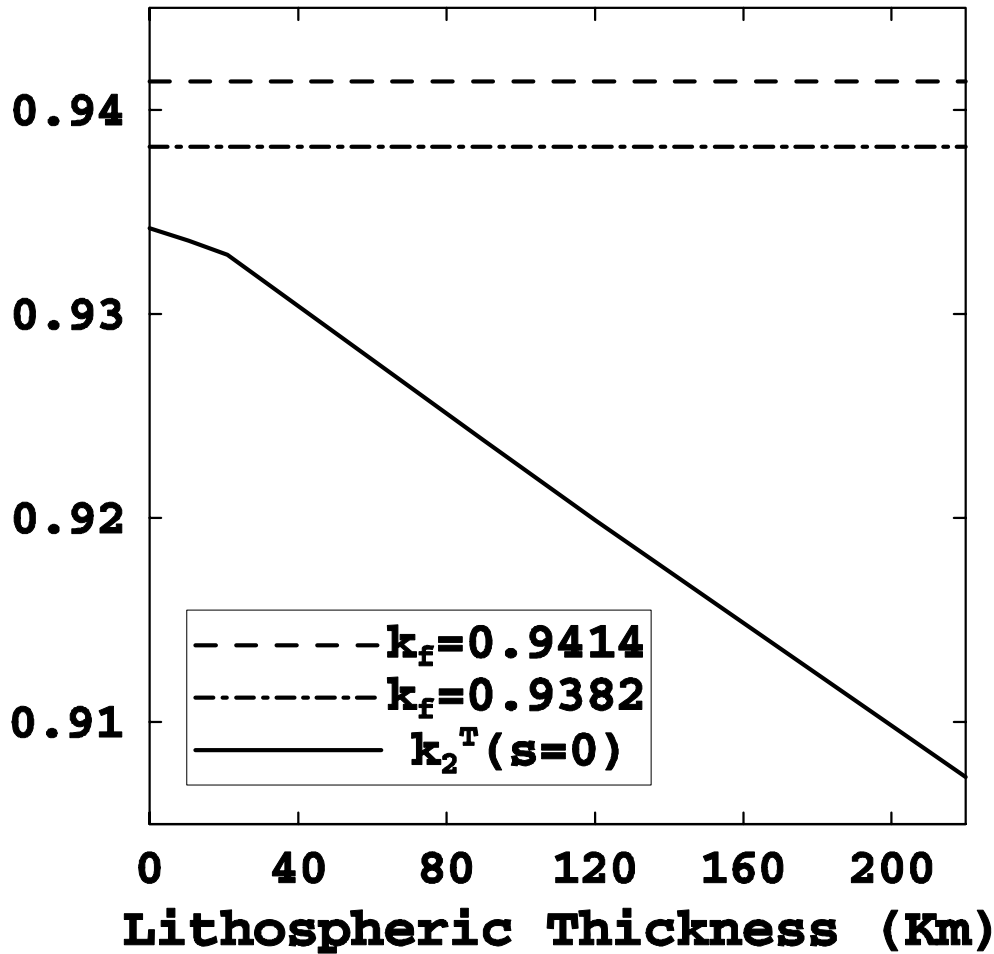
$$k_2^T(s) = k_2^T(s=0) - s \sum_{j=1}^N \frac{(q'_j/s_j)}{(s + s_j)} \quad (12a)$$

$$k_2^L(s) = (-1 + \ell_s) - s \sum_{j=1}^N \frac{(q_j/s_j)}{(s + s_j)} \quad (12b)$$

in which the superscript  $\ell=2$  on  $q_j^2, r_j^2, s_j^2$  has been suppressed for convenience. Substituting (12a) into (11a) this may be re-written as:

$$\underline{m}(s) = \frac{\underline{\Psi}^L(s)}{\left[1 - \frac{k_2^T(s=0)}{k_f}\right] + \frac{s}{k_f} \sum_{j=1}^N \frac{(q'_j / s_j)}{(s + s_j)}} \quad (13)$$

In their discussion of the formal inversion of (13) into the time domain, Peltier (1982) and Wu and Peltier (1984) made the approximation that the term in square brackets in the denominator of 13 could be safely neglected. In MW it is claimed that this renders the numerical structure employed to compute the time domain response unstable. This appears to be connected to a misunderstanding of the Tauberian Theorem (eg Widmer, 1983) which asserts that the infinite time limit of  $\mathbf{m}(t)$  will be equal to the  $s \rightarrow 0$  limit of the product  $s\mathbf{m}(s)$ . Clearly the approximation in which the square bracketed term in the denominator of (13) is neglected, in which case one is assuming that  $k_2^T(s=0) = k_f$ , the multiplication by “s” on the lhs of (13) cancels the “s” in the denominator of (13), thus rendering the infinite time limit of the approximate form of (13) entirely stable. In this brief paper my purpose is to



*Figure 1. Compares the value of the degree 2 “tidal Love number” in the limit of zero frequency ( $s=0$ ) with the two estimates of the “fluid Love number” discussed in the text*

demonstrate this fact by computing exact solutions for the inverse of (13) without making the approximation involved in the neglect of the term in square brackets in the denominator of (13). It is nevertheless useful to start this process by showing explicitly that this term is small. This is demonstrated in Figure 1 where I show  $k_2^T(s=0)$  as a function of lithospheric thickness “L”. It will be clear by inspection of this Figure, on which the two previously cited values for  $k_f$  are also shown, that in the limit of zero lithospheric thickness the approximation made in the analyses of Peltier (1982) and Wu and Peltier (1984) becomes increasingly more valid. That the Earth might be expected to respond to the GIA process such that the flattening of its figure was accurately predictable by the infinite time limit of the first order linear visco-elastic field theory of Peltier (1974) is entirely expected. The fact that it is not “exactly” predictable by this field theory (see Figure 1) is also entirely expected because processes other than the basic rotation of the object, such as mantle convection, may also contribute to this flattening. To demonstrate the impact of the approximation previously made in constructing the solutions for the polar wander speed and direction caused by the GIA process we must invert the Laplace transform domain solution (13) exactly. This was not done in MW and this appears to have clouded their judgement as to what the impact might be.

When the assumption  $k_2^T(s=0) = k_f$  is abandoned, the Laplace transform domain impulse response may then be written in the form:

$$H(s) = \left( \frac{\Omega_o}{A\sigma} \right) \frac{1 + k_2^L(s)}{\frac{s}{k_f} \sum_{j=1}^N \frac{(q_j/s_j)}{(s+s_j)} + \varepsilon} \quad (14a)$$

where

$$\varepsilon = 1 - \frac{k_2^T(s=0)}{k_f}. \quad (14b)$$

As will become clear, even though  $\varepsilon$  is a small quantity (especially in the case that the finite thickness of the lithosphere may be neglected in the limit  $t \rightarrow \infty$ ), retaining it in expression (14a) for the impulse response could have a significant impact upon the solution as the rotational stability of the system would be modified. Now the construction of the solution for the time-domain form of the impulse response  $H(t)$  proceeds in this case as in the case based upon the Equivalent Earth Model assumption, although the result differs somewhat from a physical perspective. In this case it is useful to make the distinction between the Chandler wobble frequency of a rigid Earth  $\sigma$  and the Chandler wobble frequency of the visco-elastic Earth  $\sigma_o$ , by employing the definition:

$$\sigma_o = \frac{(k_2^T(s=0) - k_2^{TE})}{k_2^T(s=0)} \sigma. \quad (15)$$

We must then re-write the expression for  $H(s)$  as:

$$H(s) = \left( \frac{\Omega}{A\sigma_o} \right) \frac{(1 + k_2^L(s))}{\left( (1 - \varepsilon)s \sum_{i=1}^N \frac{g_j}{s+s_j} \right) + \varepsilon'} \quad (16a)$$

with

$$\varepsilon' = \varepsilon \frac{\sigma}{\sigma_0}, \quad (16b)$$

and,

$$g_j = \frac{q'_j/s_j}{\sum_j (q'_j/s_j)}. \quad (16c)$$

The inversion of  $H(s)$  into the time domain now proceeds by expanding the sum in the denominator of (16a) in the form:

$$\sum_{j=1}^N \frac{g_j}{(s+s_j)} = \frac{Q_{N-1}(s)}{\prod_{j=1}^N (s+s_j)} = \frac{\prod_{j=1}^{N-1} (s+\lambda_j)}{\prod_{j=1}^N (s+s_j)} \quad (17)$$

since  $\sum_j g_j \equiv 1$ . Then we have, suppressing for the moment the factor  $(\Omega_0 / A\sigma_0)$ ,

$$H(s) = \frac{\prod_{j=1}^N (s+s_j) [1+k_2^L(s)]}{(1-\varepsilon)s \prod_{i=1}^{N-1} (s+\lambda_i) + \varepsilon' \prod_{j=1}^N (s+s_j)}. \quad (18)$$

Now substituting for the function  $1+k_2^L(s)$  from (12b) we obtain:

$$H(s) = \frac{\prod_{j=1}^N (s+s_j) \ell_s}{(1-\varepsilon)s \prod_{i=1}^{N-1} (s+\lambda_i) + \varepsilon' \prod_{i=1}^N (s+s_i)} + \sum_{j=1}^N \frac{(-q_j/s_j)s \prod_{i \neq j}^N (s+s_i)}{(1-\varepsilon)s \prod_{i=1}^{N-1} (s+\lambda_i) + \varepsilon' \prod_{i=1}^N (s+s_i)} \quad (19a)$$

$$= \frac{\prod_{j=1}^N (s+s_j) \ell_s}{(1-\varepsilon+\varepsilon') \prod_{i=1}^N (s+\kappa_i)} + \sum_{j=1}^N \frac{(-q_j/s_j)s \prod_{i \neq j}^N (s+s_i)}{(1-\varepsilon+\varepsilon') \prod_{i=1}^N (s+\kappa_i)}. \quad (19b)$$

Where now the  $\kappa_i$  are the  $N$  roots of the polynomial in the denominator of the 2 terms in (19a). This expression for the impulse response may be further reduced by re-writing the ratios of products as follows:

$$\frac{\prod_{j=1}^N (s+s_j)}{\prod_{j=1}^N (s+\kappa_j)} = 1 - \frac{q'(s)}{\prod_{i=1}^N (s+\kappa_i)} \quad (20a)$$

where now 
$$q'(s) = \prod_{j=1}^N (s+\kappa_j) - \prod_{j=1}^N (s+s_j) \quad (20b)$$



and

$$\frac{s \prod_{i \neq j}^N (s + s_i)}{\prod_{j=1}^N (s + \kappa_j)} = 1 - \frac{R'_j(s)}{\prod_{i=1}^N (s + \kappa_i)} \quad (21a)$$

with

$$R'_j(s) = \prod_{i=1}^N (s + \kappa_i) - s \prod_{i \neq j}^N (s + s_i) \quad (21b)$$

We then have, for the Laplace transform of the impulse response, the expression:

$$H(s) = \frac{\ell_s}{(1 - \varepsilon + \varepsilon')} \left\{ 1 - \frac{q'(s)}{\prod_{i=1}^N (s + \kappa_i)} \right\} + \frac{1}{(1 - \varepsilon + \varepsilon')} \sum_{j=1}^N \left( -\frac{r_j}{s_j} \right) \left\{ 1 - \frac{R'_j(s)}{\prod_{i=1}^N (s + \kappa_i)} \right\} \quad (22a)$$

or

$$H(s) = \frac{\ell_s - \sum_{j=1}^N r_j / s_j}{(1 - \varepsilon + \varepsilon')} - \frac{\ell_s q'(s)}{(1 - \varepsilon + \varepsilon') \prod_{i=1}^N (s + \kappa_i)} + \frac{1}{(1 - \varepsilon + \varepsilon')} \sum_{j=1}^N \frac{(q_j / s_j) R'_j(s)}{\prod_{i=1}^N (s + \kappa_i)} \quad (22b)$$

Denoting  $\ell_s - \sum_{j=1}^N r_j / s_j = 1 + k_2^{LE} = D_1$ , say, then we may further reduce the expression for the impulse response to:

$$H(s) = \frac{D_1}{(1 - \varepsilon + \varepsilon')} - \frac{1}{(1 - \varepsilon + \varepsilon')} \left\{ \frac{\ell_s q'(s) - \sum_{j=1}^N (q_j / s_j) R'_j(s)}{\prod_{i=1}^N (s + \kappa_i)} \right\} \quad (23)$$

The inverse Laplace transform of this expression is such that the solution in the present case, in which  $k_2^T(s=0) \neq k_f$ , is just:

$$m_1(t) = \frac{1}{(1 - \varepsilon + \varepsilon')} \left( \frac{\Omega_o}{A \sigma_o} \right) \left\{ \left[ \ell_s - \sum_{j=1}^N \frac{r_j}{s_j} \right] I_{13}^{\text{Rigid}}(t) + \sum_{i=1}^N E'_i e^{-\kappa_i t} * I_{13}^{\text{Rigid}}(t) \right\} \quad (24a)$$

$$m_2(t) = \frac{1}{(1 - \varepsilon + \varepsilon')} \left( \frac{\Omega_o}{A \sigma_o} \right) \left\{ \left[ \ell_s - \sum_{j=1}^N \frac{r_j}{s_j} \right] I_{23}^{\text{Rigid}}(t) + \sum_{i=1}^N E'_i e^{-\kappa_i t} * I_{23}^{\text{Rigid}}(t) \right\} \quad (24b)$$

where

$$E'_i = \left\{ -\ell_s q'(-\kappa_i) + \sum_{j=1}^N \frac{r_j}{s_j} R'_j(-\kappa_i) \right\} / \prod_{i=j}^N (\kappa_j - \kappa_i). \quad (24c)$$

The polar wander velocity vector components are obtained simply by time differentiation of equations (24a) and (24b). It is useful to compare the result in (24) to the solutions that obtain under the approximation previously employed. In the limit  $\varepsilon \rightarrow 0$  we have  $\kappa_N = 0$  and  $\kappa_i = \lambda_i$  the N-1 relaxation times that govern the system in this limit. In this case, the parameter  $E'_N$  in the above becomes:

$$E'_N = -\frac{\ell_1 q(0)}{\prod_{j=1}^{N-1} (\kappa_j - \kappa_N)} = -\frac{\ell_s q(0)}{\prod_{j=1}^{N-1} \lambda_j} \quad (25)$$

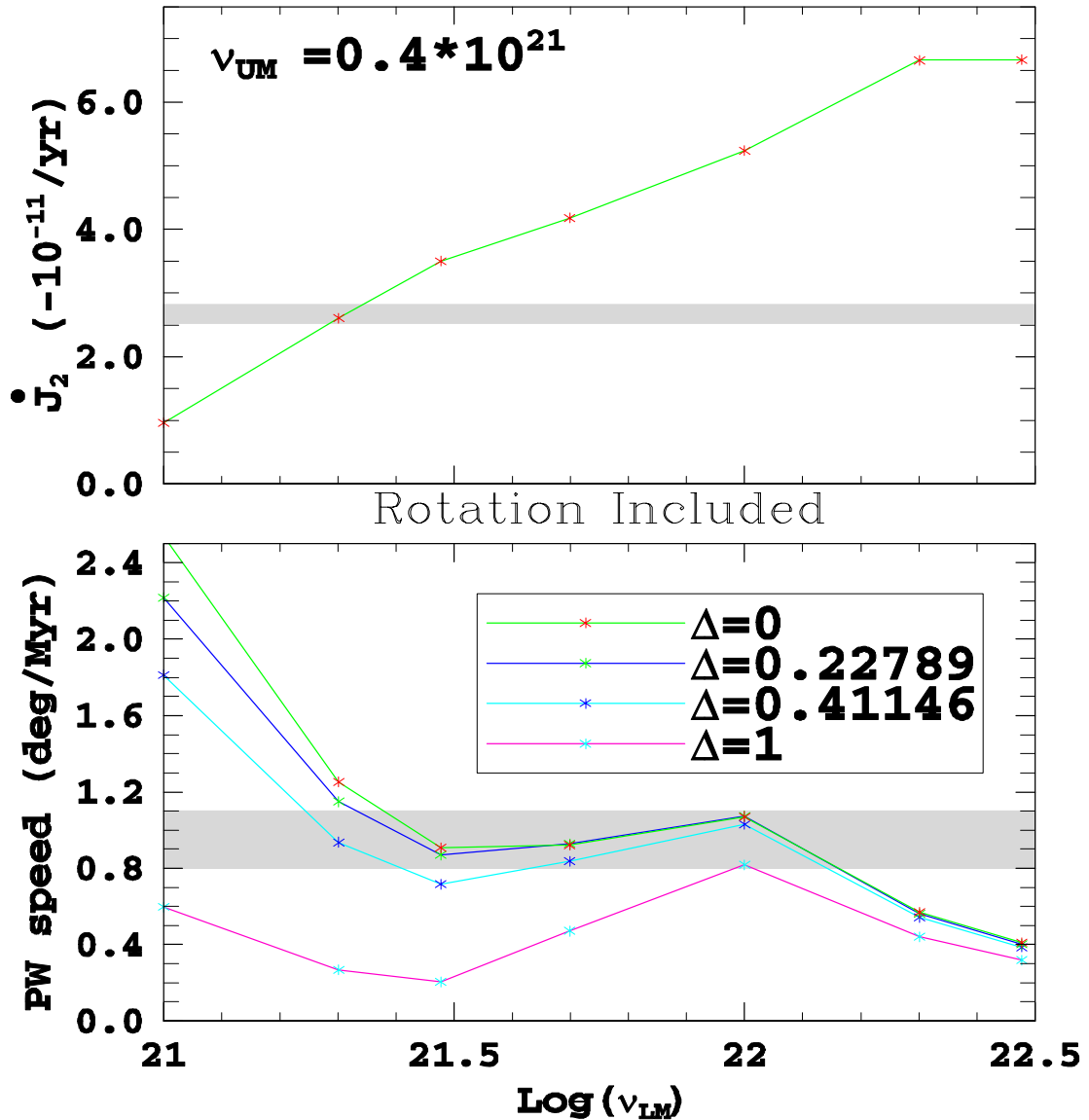
And the previous approximate result is fully recovered.

In order to compare the temporal histories of the rotational anomalies in the two cases, it will be important to proceed by keeping as many features of the Earth model fixed as possible. To this end and for the remainder of this paper, I will focus entirely upon the nature of the solutions that obtain when the recently published ICE-5G model of the glaciation and deglaciation process of Peltier (2004) is employed to determine the rotational excitation functions required for the evaluation of the solution (24). In the next section results will be discussed for a sequence of simple two layered viscosity structures as a function of the parameter  $\varepsilon$  in order to explicitly demonstrate the highly stable nature of the solution in the limit that this parameter vanishes.

## Results

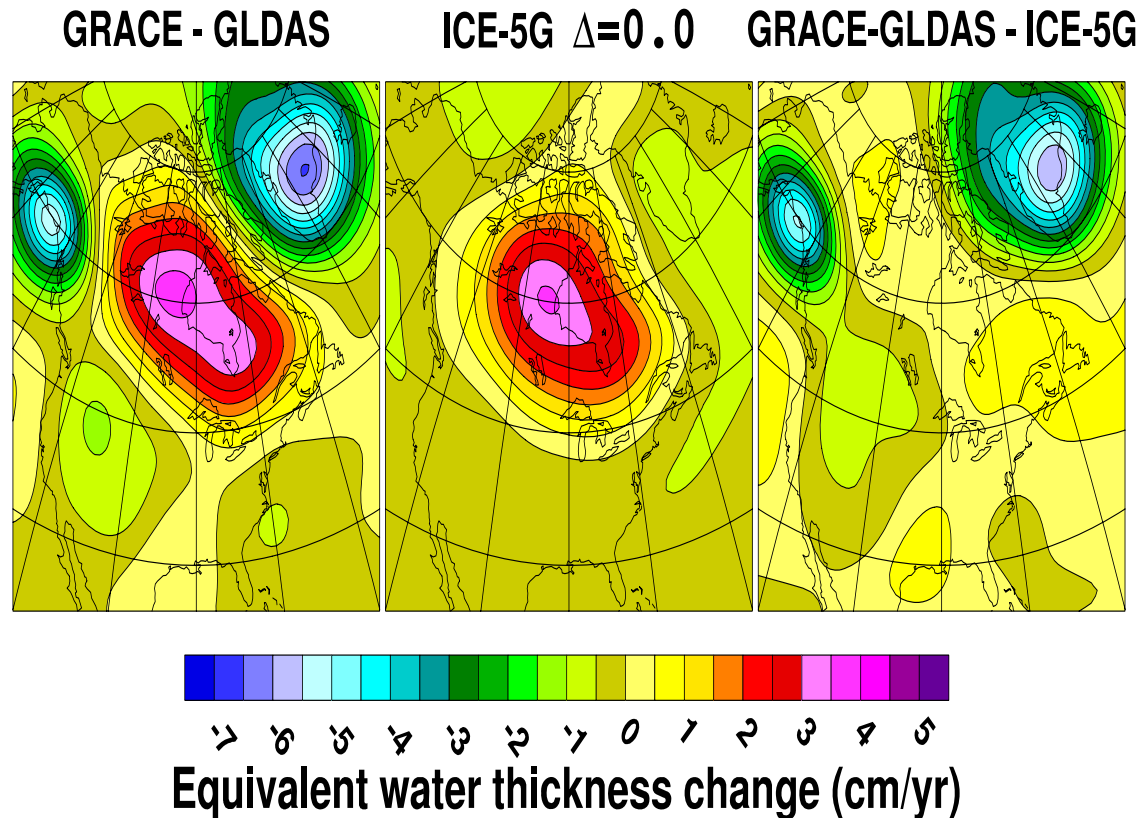
Of particular importance for the purpose of this paper is the sensitivity of the predictions of polar wander speed to the assumption that  $k_2^T(s=0)$  may be assumed to be equal to  $k_f$ . When this assumption is not made, then the solution is given by equation (24). In the latter, there appears the quantity  $(1 - \varepsilon + \varepsilon')$ , the values in which for the Earth model (VM2) in question are respectively 0.034, 0.05, and 1.017 (for  $\varepsilon$ ,  $\varepsilon'$  and  $1 - \varepsilon + \varepsilon'$ ) when the thickness of the lithosphere is taken to be 90 km. In Figure 2 (bottom) are plotted the predictions of polar wander speed based upon equations (24) as a function of the viscosity of the lower mantle with the upper mantle viscosity held fixed to the value in the VM2 model of Peltier (1996). Results are also shown for several different values of a parameter  $\Delta = \varepsilon / 0.034$  including the value  $\varepsilon = 0.034$  ( $\Delta = 1$ ) which is appropriate for the VM2 model with a lithospheric thickness of 90 km, in which case  $k_2^T(s=0) = 0.9263$ , but also for significantly smaller values of  $\varepsilon$  including the value  $\varepsilon = 0$  ( $\Delta = 0$ ) so as to investigate the “smoothness” of the transition from the value  $\varepsilon = 0$  which obtains when  $k_2^T(s=0)$  is assumed to be equal to  $k_f$ . The two intermediate values of  $\Delta$  for which results are shown on Figure 2 correspond to the two values of  $k_f$  shown on Figure 1 when the lithospheric thickness  $L$  is assumed to be equal to zero. Also shown on Figure 2 (top) is the dependence of the predicted value of the non-tidal acceleration as a function of lower mantle viscosity.

Inspection of Figure 2 clearly demonstrates the fact that the solutions for polar wander speed that obtain in the limit  $\Delta = 0$  are almost identical to those that obtain for either of the two non-zero values that correspond to zero lithospheric thickness. This demonstrates that the formulation of Peltier (1982) and Wu and Peltier (1984) based upon the approximation  $k_2^T(s=0) = k_f$  was not mathematically unstable as claimed in WM. In fact, careful inspection of Figure 2 will show that the preferred solution for BOTH the non-tidal acceleration and polar wander speed is the model



**Figure 2.** This Figure compares model predictions of the non-tidal acceleration of rotation (top) and of the speed of polar wander (bottom) as a function of the viscosity of the lower mantle when the upper mantle viscosity is held fixed to the value in the VM2 viscosity model of Peltier (1996). The polar wander speed predictions are shown for several values of the parameter  $\Delta$  which measures the importance of the difference between the fluid Love number  $k_f$  and  $k_2^T(s=0)$ . The two values of  $\Delta$  that are less than unity, 0.22789 and 0.41146, correspond respectively to the  $k_f$  values of 0.9382 and 0.9414 and are those that obtain in the limit of vanishing lithospheric thickness. The value  $\Delta = 1$  is the value appropriate for a finite lithospheric thickness of 90 km.

with  $\Delta = 0.41146$  AND  $L=0.0$ . This solution amounts to a very modest adjustment of the earlier result obtained with  $\Delta = 0.0$  and  $L=0.0$ . The results for finite non-zero lithospheric thickness cannot fit the observed polar wander speed except, marginally, for a model with an upper mantle-lower mantle viscosity contrast that is incompatible with the observed non-tidal acceleration. Such high contrast viscosity models are also firmly rejected by relative sea level data from the previously ice covered area of North America.



*Figure 3. Demonstrates the ability of the GIA model of Peltier(2004) to accurately explain the observed time dependence of the gravity field over the North American continent. This field is represented by the time rate of change of the thickness of an equivalent layer of water at the earth's surface. This analysis is based upon the level 2 release of the GRACE Stokes coefficients. In this comparison, the degree 2 terms have been excluded, a consequence of the fact that GRACE does not provide accurate measures of these coefficients.*

The quality of this low contrast model is also strongly re-enforced by the recently obtained time dependent gravity field data from the GRACE satellite system. Figure 3 compares the GRACE observed and hydrology corrected GRACE time dependent gravity field observations with the ICE-5G(VM2) GIA model prediction of the same field. In the third frame of Figure 3 the difference between these two data sets is also shown, thus demonstrating the extremely high quality of the ICE-5G(VM2) model. The neglect of the degree 2 coefficients, which are very large for the ICE-5G(VM2) model, as demonstrated in Peltier (2004), is required by virtue of the inability of GRACE to accurately observe these coefficients..

**Conclusion**

The analyses described in the previous sections of this paper have considerably extended the previously published theory that is employed to compute the response of

the earth's rotational state to the global process of glacial isostatic adjustment. These analyses suffice to refute the claim in MW that the formalism described in Peltier (1982) and Wu and Peltier (1984) was fundamentally unstable mathematically. This error of interpretation appears to have been due to a lack of understanding of the Tauberian Theorem that may be employed to predict the infinite time limit of a solution from the Laplace transform of this solution. The extended version of the theory described herein has allowed a direct investigation of the question of the extent to which the finite thickness of a globally continuous and unbroken lithosphere may contribute to the rotational response to surface mass load forcing. These analyses demonstrate that, in this long timescale limit, the most accurate representation of the rotational response of the Earth is that based upon the assumption of vanishing lithospheric thickness. This is understandable on the basis of the fact that the lithosphere of the planet is "broken" into a series of weakly coupled plates. For planets whose lithospheres are not unbroken in this way, the same assumption would clearly not be appropriate.

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